Large sample variances of maximum likelihood estimators of variance components.

S. R. Searle

Texas A & M University*

Summary

A general expression is obtained for the elements of the information matrix of the maximum likelihood estimators of variance components derived from unbalanced data of any mixed model. This general expression is used to obtain explicit results for the 2-way nested classification, random model.

Introduction

Estimation of variance components from data that are unbalanced (having unequal numbers of observations in the subclasses) has been referred to by Hartley (1967) as involving "algebraic heroics". Obtaining sampling variances can be described similarly. There is little or no difficulty with balanced data because then, with customary normality assumptions, the sums of squares of an analysis of variance have independent χ^2-distributions, variance component estimators are linear functions of these and their variances are readily derived. Furthermore, as Scheffé (1959) points out, these estimators (which are unbiased) are also minimum variance.

In contrast to the relatively manageable situation of balanced data we consider here the estimation of variance components from unbalanced data, of which balanced data are deemed to be simply a special case. In particular we deal with the sampling variances of large sample maximum likelihood estimators. Before doing so, a brief review of the present status of sampling variances of other estimators is in order.

For random (and mixed) models Henderson (1953) developed three methods of estimating variance components from unbalanced data of any crossed and/or nested

---

* On leave, 1968-9, from the Biometrics Unit, Cornell University, Ithaca, New York.

classifications, the techniques of which have been further discussed in Searle (1968). Sampling variances of the resulting estimators have been extensively developed for only one of these methods, Method 1, that which is analogous to the analysis of variance method for balanced data. Furthermore, in all cases, sampling variances have been considered only in the case of random effects models [Model II of Eisenhart (1947)], and only on the basis of having normality assumptions as part of such models. Within this framework Crump (1951) derived sampling variances for variance components estimators obtained from 1-way classification data (between and within subclasses). Searle (1956), using matrix methods, re-worked Crump's results and extended them to components of covariance; with the same methods Searle (1958 and 1961) derived sampling variances of variance components estimators for both the 2-way crossed and the 2-way nested classifications. Mahamunulu (1953) extended the methods to the 3-way nested classification, and Blischke (1966 and 1968) has used them on a 3-way crossed classification mixed model and on r-way classifications in general. These variances can also be found, in specific cases, by means of a (largely computational) method developed by Hartley (1967), a method which has been extended by Rao (1968).

Maximum likelihood estimation of variance components from unbalanced data has received somewhat less attention than other methods of estimation due to its underlying complexities. One of these is the need for maximizing over only non-negative values of the components, a difficulty that has been dealt with by Herbach (1959) and Thompson (1961) for balanced data. With unbalanced data, however, the maximizing equations appear to yield no explicit solutions for the estimators. For example, even in the simplest case, that of the 1-way classification, terms such as $\sum_i n_i/(\sigma_e^2 + n_i\sigma_\alpha^2)$ are involved in equations that have to be solved for σ_e^2 and σ_α^2. In view of this apparent intractability, Hartley and Rao (1967) have considered iterative procedures for solving such equations, and in doing so they handle the quite general case of unbalanced data from any mixed (or random) model.

Despite the inability to obtain explicit expressions for maximum likelihood estimators of variance components derived from unbalanced data, Crump (1951) obtained their large-sample variances in the case of the 1-way classification. These results we now extend. In general the large-sample variance-covariance matrix of maximum likelihood estimators of parameters of any model is the inverse of the information matrix. And this matrix is minus the expected value of the Hessian with respect to those parameters of the logarithm of the likelihood. A general expression for an element of this matrix is here developed, apropos unbalanced data of any mixed (or random) effects model. This general result is then used to derive explicit expressions appropriate to variance components estimators in the 2-way nested classification random model.

One value of the resulting expressions is that even though maximum likelihood estimators of the σ^2's cannot be obtained, values for their variances can; and against these can be compared variances of estimators obtained by other methods to give measures of efficiency of those other methods.

The General Linear Model - Mixed Model

The model

Consider the general linear model written as $\underline{y} = \mu\underline{1} + \underline{W}\underline{\gamma} + \underline{e}$, where $\underline{y}$ is a vector of N responses, μ is a general mean, $\underline{1}$ is a vector of N 1's, $\underline{W}$ is a known matrix, $\underline{\gamma}$ is a vector of parameters (effects) in the model and $\underline{e}$ is a vector of error terms. If all the effects in $\underline{\gamma}$ are fixed effects the model is a fixed effects model; if they are all random it is a random effects model; and if some are fixed and some are random the model is known as a mixed model. In greatest generality, however, all models of the above form can be considered as mixed models because, regardless of the elements of $\underline{\gamma}$, the general mean μ is a fixed effect and the error terms in $\underline{e}$ are random. Thus, without loss of generality, any linear model can be considered as a mixed model and expressed as

$$\underline{y} = \underline{X\beta} + \underline{Zu} \tag{1}$$

where $\underline{y}$ is a vector of N observations; $\underline{\beta}$ is a p x 1 vector of fixed effects, that include the general mean μ; $\underline{u}$ is a vector of random effects, that include the error terms $\underline{e}$; and $\underline{X}$ and $\underline{Z}$ are known matrices--often, but not always, design matrices. The random effects are further specified as having zero mean and a variance-covariance matrix $\underline{A}$, that involves q variance components $\sigma_1^2, \ldots, \sigma_q^2$, one of which is the error variance. In addition, for purposes of using maximum likelihood we adopt normality assumptions and so have $\underline{u}$ distributed as $N(\underline{0}, \underline{A})$. Thus $\underline{y}$ has variance-covariance

$$\text{var}(\underline{y}) = \text{var}(\underline{Zu}) = \underline{ZAZ}' = \underline{V} \text{ say,} \tag{2}$$

and so $\underline{y}$ is distributed as $N(\underline{X\beta}, \underline{V})$ where elements of $\underline{V}$ are linear functions of the q variance components.

For the above model the likelihood of the sample is

$$(2\pi)^{-\frac{1}{2}N} \, |\underline{V}|^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}(\underline{y} - \underline{X\beta})'\underline{V}^{-1}(\underline{y} - \underline{X\beta})\right]$$

and, apart from a constant, the logarithm of this is

$$L = -\tfrac{1}{2}\log|\underline{V}| - \tfrac{1}{2}(\underline{y} - \underline{X\beta})'\underline{V}^{-1}(\underline{y} - \underline{X\beta}) \; .$$

<u>The information matrix</u>

The variance-covariance matrix of the large sample maximum likelihood estimators of the p elements of β and the q variance components is the inverse of the information matrix, which is minus the expected value of the Hessian of L with respect to these p + q parameters. The sub-matrices of this Hessian are:

$\underline{L}_{\beta\beta}$, a pxp matrix of elements $\dfrac{\partial^2 L}{\partial\beta_h \; \partial\beta_k}$ for h,k = 1, ..., p;

$\underline{L}_{\beta\sigma^2}$, a pxq matrix of elements $\dfrac{\partial^2 L}{\partial\beta_h \; \partial\sigma_j^2}$ for h = 1, ..., p and j = 1, ..., q;

and $\underline{L}_{\sigma^2\sigma^2}$, a qxq matrix of elements $\dfrac{\partial^2 L}{\partial\sigma_i^2 \; \partial\sigma_j^2}$ for i,j = 1, ..., q.

Then the variance-covariance matrix we seek is

$$\underline{V}_{ML} = \begin{bmatrix} \mathrm{var}(\tilde{\underline{\beta}}) & \mathrm{cov}(\tilde{\underline{\beta}}\tilde{\underline{\sigma}}^2) \\ \mathrm{cov}(\tilde{\underline{\sigma}}^2\tilde{\underline{\beta}}) & \mathrm{var}(\tilde{\underline{\sigma}}^2) \end{bmatrix} = \begin{bmatrix} -E(\underline{L}_{\beta\beta}) & -E(\underline{L}_{\beta\sigma^2}) \\ -E(\underline{L}_{\beta\sigma^2})' & -E(\underline{L}_{\sigma^2\sigma^2}) \end{bmatrix}^{-1} \qquad (3)$$

where $\tilde{\underline{\beta}}$ is the M.L. estimator of $\underline{\beta}$ and $\tilde{\underline{\sigma}}^2$ is the vector of M.L. estimators of the q variance components.

For convenience write

$$d = \log|\underline{V}|$$

Then

$$L = -\tfrac{1}{2}d - \tfrac{1}{2}(\underline{y} - \underline{X\beta})'\underline{V}^{-1}(\underline{y} - \underline{X\beta})$$

and

$$\underline{L}_{\beta\beta} = \underline{X}'\underline{V}^{-1}\underline{X}$$

$$\underline{L}_{\beta\sigma^2} = \left\{\underline{X}'(\underline{V}^{-1})_j(\underline{y} - \underline{X\beta})\right\} \text{ for } j = 1,2, \ldots, q;$$

and $\underline{L}_{\sigma^2\sigma^2} = \left\{-\tfrac{1}{2}(d)_{ij} - \tfrac{1}{2}(\underline{y} - \underline{X\beta})'(\underline{V}^{-1})_{ij}(\underline{y} - \underline{X\beta})\right\}$ for i,j = 1,2, ..., q.

In these expressions $(d)_{ij}$ and $(\underline{V}^{-1})_{ij}$ are partial derivatives of $d = \log|\underline{V}|$ and $\underline{V}^{-1}$ with respect to σ_i^2 and σ_j^2. Thus

$$d_{ij} = \frac{\partial^2 d}{\partial\sigma_i^2\,\partial\sigma_j^2} = \frac{\partial^2 \log|\underline{V}|}{\partial\sigma_i^2\,\partial\sigma_j^2}$$

and $(\underline{V}^{-1})_{ij} = \dfrac{\partial^2\underline{V}^{-1}}{\partial\sigma_i^2\,\partial\sigma_j^2} = \left\{\dfrac{\partial v^{r,s}}{\partial\sigma_i^2\,\partial\sigma_j^2}\right\}$ for r,s = 1, 2, ..., N,

where $v^{r,s}$ is the (r,s)'th element in $\underline{V}^{-1}$; i.e., $(\underline{V}^{-1})_{ij}$ is the matrix $\underline{V}^{-1}$ with every element differentiated with respect to σ_i^2 and σ_j^2, for i and j taking values 1,2, ..., q.

Taking expectations we now have

$$E(\underline{L}_{\beta\beta}) = \underline{X}\underline{V}^{-1}\underline{X}$$

$$E(\underline{L}_{\beta\sigma^2}) = \left\{\underline{X}'(\underline{V}^{-1})_j E(\underline{y} - \underline{X\beta})\right\}$$

$$= 0, \quad \text{for } j = 1,2, \ldots, q;$$

and $$E(\underline{L}_{\sigma^2\sigma^2}) = \left\{-\tfrac{1}{2}(d)_{ij} - \tfrac{1}{2}\mathrm{tr}\left[E(\underline{y} - \underline{X\beta})\,(\underline{y} - \underline{X\beta})'(\underline{V}^{-1})_{ij}\right]\right\}$$

$$= \left\{-\tfrac{1}{2}(d)_{ij} - \tfrac{1}{2}\mathrm{tr}\left[\underline{V}(\underline{V}^{-1})_{ij}\right]\right\} \quad \text{for } i,j = 1,2, \ldots, q.$$

Substituting these expected values into (3) gives

$$\underline{V}_{ML} = \begin{bmatrix} \underline{X}'\underline{V}^{-1}\underline{X} & \underline{0} \\ \underline{0} & \tfrac{1}{2}\left\{(\log|V|)_{ij} + \mathrm{tr}\left[\underline{V}(\underline{V}^{-1})_{ij}\right]\right\} \quad i,j = 1, \ldots, q \end{bmatrix}^{-1} \tag{4}$$

<u>Variance-covariance matrices</u>

Several features of this result deserve mention. The first is that

$$\mathrm{cov}(\underline{\tilde{\beta}\tilde{\sigma}^2}) = \underline{0};$$

i.e., covariances between large sample M.L. estimators of fixed effects and variance components are zero. Since under conditions of normality the mean of a sample and its sum of squares are independent, this result is not surprising.

A second consequence of (4) is that the variance-covariance matrix of the large sample M.L. estimators of the fixed effects is

$$\mathrm{var}(\tilde{\underline{\beta}}) = (\underline{X}'\underline{V}^{-1}\underline{X})^{-1} \ . \qquad (5)$$

This corresponds to the equations whose solutions for $\tilde{\underline{\beta}}$ are the M.L. estimators of $\underline{\beta}$, namely $\underline{L}_\beta = 0$, which is

$$\underline{X}'\tilde{\underline{V}}^{-1}\underline{X}\tilde{\underline{\beta}} = \underline{X}'\tilde{\underline{V}}^{-1}\underline{y} \qquad (6)$$

where $\tilde{\underline{V}}^{-1}$ is $\underline{V}^{-1}$ with the variance components $\sigma_1^2, \ldots, \sigma_q^2$ replaced by their M.L. estimators. This equation, (6), corresponds to (10) of Hartley and Rao (1967). Result (5) is also what one would get from estimating $\underline{\beta}$ by least squares from the model (1) assuming $\underline{V}$ of (2) to be known. It is therefore no surprise to see it arising from maximum likelihood. Nevertheless, it is interesting to note from this that even with unbalanced data from <u>any</u> mixed model, the variance matrix of the M.L. estimators of the fixed effects is exactly as it would be when the variance components were known and not having to be estimated simultaneously with the fixed effects.

Finally from (3) and (4) the variance-covariance matrix of the large sample M.L. estimators of the variance components is

$$\mathrm{var}(\tilde{\underline{\sigma}}^2) = 2\left\{(\log|V|)_{ij} + \mathrm{tr}\left[\underline{V}(\underline{V}^{-1})_{ij}\right] \text{ for } i,j = 1,2, \ldots, q\right\}^{-1} . \qquad (7)$$

This, we see, is entirely free of the fixed effects and is solely a function of the variance-covariance matrix $\underline{V}$ of the observations $\underline{y}$. Since $\underline{V}$ in turn depends only on the occurrence of the random effects in the model, we see that no matter what the fixed effects are, nor how they occur, they in no way affect $\mathrm{var}(\tilde{\underline{\sigma}}^2)$. Only to the extent that the fixed effects and random effects interact, with interaction effects that are deemed random - only to this extent do fixed effects affect $\mathrm{var}(\tilde{\underline{\sigma}}^2)$.

Further simplification of (9) comes not from substituting $\underline{V} = \underline{ZAZ}'$ from (2) but from writing (7) as

$$\mathrm{var}(\underline{\tilde{\sigma}}^2) = 2\underline{T}^{-1} = 2\left\{t_{ij}\right\}^{-1} \tag{8}$$

with

$$t_{ij} = (\log|\underline{V}|)_{ij} + \mathrm{tr}\left[\underline{V}(\underline{V}^{-1})_{ij}\right] \tag{9}$$

where the subscripts on the right-hand side denote partial differentiation with respect to σ_i^2 and σ_j^2, for $i,j = 1,2, \ldots, q$. To simplify we use a result implied in Hartley and Rao (1967) that

$$\frac{\partial}{\partial\sigma_i^2}\log|\underline{V}| = \mathrm{tr}(\underline{V}^{-1}\frac{\partial}{\partial\sigma_i^2}\underline{V}) ,$$

i.e. in our notation that

$$(\log|\underline{V}|)_i = \mathrm{tr}(\underline{V}^{-1}\underline{V}_i) . \tag{10}$$

Hence, because

$$(\underline{V}^{-1})_i = -\underline{V}^{-1}\underline{V}_i\underline{V}^{-1} , \tag{11}$$

differentiation of (10) and (11) with respect to σ_j^2 gives

$$(\log|\underline{V}|)_{ij} = \mathrm{tr}(-\underline{V}^{-1}\underline{V}_j\underline{V}^{-1}\underline{V}_i + \underline{V}^{-1}\underline{V}_{ij}) \tag{12}$$

and

$$(\underline{V}^{-1})_{ij} = \underline{V}^{-1}\underline{V}_j\underline{V}^{-1}\underline{V}_i\underline{V}^{-1} - \underline{V}^{-1}\underline{V}_{ij}\underline{V}^{-1} + \underline{V}^{-1}\underline{V}_i\underline{V}^{-1}\underline{V}_j\underline{V}^{-1} .$$

Multiplying this by $\underline{V}$ and taking the trace gives

$$\mathrm{tr}\left[\underline{V}(\underline{V}^{-1})_{ij}\right] = \mathrm{tr}(\underline{V}_j\underline{V}^{-1}\underline{V}_i\underline{V}^{-1} - \underline{V}_{ij}\underline{V}^{-1} + \underline{V}_i\underline{V}^{-1}\underline{V}_j\underline{V}^{-1})$$

$$= \mathrm{tr}(2\underline{V}^{-1}\underline{V}_i\underline{V}^{-1}\underline{V}_j - \underline{V}^{-1}\underline{V}_{ij}) ,$$

and substituting this and (12) into (9) gives

$$t_{ij} = \mathrm{tr}(\underline{V}^{-1}\underline{V}_i\underline{V}^{-1}\underline{V}_j) ,$$

and so in (8)

$$\mathrm{var}(\underline{\tilde{\sigma}}^2) = 2\underline{T}^{-1} = 2\left\{\mathrm{tr}(\underline{V}^{-1}\underline{V}_i\underline{V}^{-1}\underline{V}_j)\right\}^{-1} \text{ for } i = 1,2, \ldots, q \quad (13)$$

where $\underline{V}_i$ and $\underline{V}_j$ are the partial differentials of $\underline{V}$ with respect to σ_i^2 and σ_j^2. It is worth emphasizing that in this formulation $\underline{V}$ is the variance-covariance matrix of the vector y and as such includes σ_e^2 as well as the variances of the random effect.

A simple example

Before proceeding to use (13) for the 2-way nested classification random model, consider as a simple example of its use the model

$$y_i = \mu + e_i \text{ for } i = 1,2, \ldots, N \text{ with } \underline{e} \sim N(0, \sigma^2\underline{I}_N).$$

Then $\underline{V} = \sigma^2\underline{I}_N$, $\underline{V}^{-1} = (1/\sigma^2)\underline{I}_N$ and $\underline{V}_{\sigma^2} = \underline{I}_N$. Hence

$$\begin{aligned} \mathrm{var}(\tilde{\sigma}^2) &= 2\left\{\mathrm{tr}\left[(1/\sigma^2)\underline{I}_N\underline{I}_N\right]^2\right\}^{-1} \\ &= 2(N/\sigma^4)^{-1} \\ &= 2\sigma^4/N , \end{aligned}$$

as is to be expected.

Estimating the mean of a random model

A special case of (6) is in the random effects model where the fixed effects terms $\underline{X}\underline{\beta}$ in the general model (1) are $\mu\underline{1}$. The M.L. estimator of μ is then

$$\tilde{\mu} = \underline{1}'\tilde{\underline{V}}^{-1}\underline{y}/\underline{1}'\tilde{\underline{V}}^{-1}\underline{1} \quad (14)$$

as mentioned by Koch (1967), its denominator being the sum of all elements in $\tilde{\underline{V}}^{-1}$; and the variance of this estimator is, from (5),

$$\mathrm{var}(\tilde{\mu}) = 1/\underline{1}'\underline{V}^{-1}\underline{1} , \quad (15)$$

the reciprocal of the sum of all elements of $\underline{V}^{-1}$.

The 2-Way Nested Classification

The model

The random model for the 2-way nested classification can be written as

$$y_{ijk} = \mu + \alpha_i + \beta_{ij} + e_{ijk} \tag{16}$$

where y_{ijk} is the k'th response in the j'th level of the β-effects in the i'th level of the α-effects. μ is a general mean, α_i is the effect due to the i'th level of the α-effects, the main classification, and β_{ij} is the effect due to the j'th level of the β-effects within the i'th level of the α's, and e_{ijk} is the random error term peculiar to y_{ijk}. It is assumed that the α's, β's and e's are all normally and independently distributed, with zero means and variances σ_α^2, σ_β^2 and σ_e^2, respectively.

The number of levels of the α-effects is denoted by a, so that i = 1,2, ..., a; and the number of levels of β-effects within the i'th α-effect is c_i, so that j = 1,2, ..., c_i. The total number of such sub-classes will be represented by b, giving $b = \sum_{i=1}^{a} c_i = c_.$; and the number of observations in the j'th sub-class of the i'th class is taken as n_{ij}, so that k = 1,2, ..., n_{ij}.

An example

Suppose that for 2 levels of the α-effects we have the following data

Table 1. Example of data

Levels of α_i	Observations and n_{ij}'s			$n_{i.}$	c_i
	Levels of β_{ij} within α_i				
	1	2	3		
1	y_{111}, y_{112} (n_{11}=2)	$y_{121}, y_{122}, y_{123}$ (n_{12}=3)	-	$n_{1.}$=5	c_1=2
2	y_{211} (n_{21}=1)	y_{221}, y_{222} (n_{22}=2)	y_{231} (n_{23}=1)	$n_{2.}$=4	
a = 2				$n_{..}$=9	$c_.$=5

On defining $\underline{y}$ as the vector of observations arrayed in k-order, within j-order within i, we have

$$\underline{y}' = \left(y_{111},\ y_{112},\ y_{121},\ y_{122},\ y_{123},\ y_{211},\ y_{221},\ y_{222},\ y_{231}\right).$$

To simplify notation in the writing of $\underline{V} = \mathrm{var}(\underline{y})$, we use α and σ^2_α, β and σ^2_β, and e and σ^2_e interchangeably; i.e. $\alpha \equiv \sigma^2_\alpha$, $\beta \equiv \sigma^2_\beta$ and $e \equiv \sigma^2_e$. Then the value of $\underline{V}$ for this example is

$$\underline{V} = \left[\begin{array}{ccccccccc}
\alpha+\beta+e & \alpha+\beta & \alpha & \alpha & \alpha & & & & \\
\alpha+\beta & \alpha+\beta+e & \alpha & \alpha & \alpha & & & & \\
\alpha & \alpha & \alpha+\beta+e & \alpha+\beta & \alpha+\beta & & \underline{0} & & \\
\alpha & \alpha & \alpha+\beta & \alpha+\beta+e & \alpha+\beta & & & & \\
\alpha & \alpha & \alpha+\beta & \alpha+\beta & \alpha+\beta+e & & & & \\
 & \underline{0} & & & & \alpha+\beta+e & \alpha & \alpha & \alpha \\
 & & & & & \alpha & \alpha+\beta+e & \alpha+\beta & \alpha \\
 & & & & & \alpha & \alpha+\beta & \alpha+\beta+e & \alpha \\
 & & & & & \alpha & \alpha & \alpha & \alpha+\beta+e
\end{array}\right] \tag{17}$$

The matrix V and its inverse

From the above example the general form of $\underline{V}$ is clear. It is a diagonal matrix of square submatrices $\underline{V}_i$ of order $n_{i.}$ for i = 1,2, ..., a:

$$\underline{V} = \begin{bmatrix} \underline{V}_1 & 0 & \cdots & 0 \\ 0 & \underline{V}_2 & & \\ & & & 0 \\ 0 & & 0 & \underline{V}_a \end{bmatrix} \tag{18}$$

Thus $\underline{V}$ is the direct sum of the matrices $\underline{V}_i$ and so we write

$$\underline{V} = \sum_{i=1}^{a}{}^{+}\ \underline{V}_i \tag{19}$$

using the symbol Σ^+ to denote the operation of direct sum. On comparing (17) and (18) it can be seen that $\underline{V}_i$ may be partitioned as

$$V_i = \left\{\underline{V}_{i,jj'}\right\} \text{ for } j,j' = 1,2, \ldots, c_i \tag{20}$$

where

$$\underline{V}_{i,jj} = e\underline{I}_{n_{ij}} + (\alpha+\beta)\, \underline{J}_{n_{ij}} \tag{21}$$

and

$$\underline{V}_{i,jj'} = \alpha \underline{J}_{n_{ij} \times n_{ij'}} \quad \text{for } j \neq j'. \tag{22}$$

where $\underline{J}_{n_{ij} \times n_{ij'}}$ is a matrix of order $n_{ij} \times n_{ij'}$ with every element unity; and $\underline{J}_{n_{ij}}$ is $\underline{J}_{n_{ij} \times n_{ij}}$, square of order n_{ij}.

<u>Sampling variances</u>

The variance-covariance matrix of the large sample M.L. estimators of $\sigma^2_\alpha \equiv \alpha$, $\sigma^2_\beta \equiv \beta$ and $\sigma^2_e \equiv e$ that we seek is, as in (3),

$$V_{ML} = \begin{bmatrix} \text{var}(\tilde{\sigma}^2_\alpha) & \text{cov}(\tilde{\sigma}^2_\alpha, \tilde{\sigma}^2_\beta) & \text{cov}(\tilde{\sigma}^2_\alpha, \tilde{\sigma}^2_e) \\ \text{cov}(\tilde{\sigma}^2_\alpha, \tilde{\sigma}^2_\beta) & \text{var}(\tilde{\sigma}^2_\beta) & \text{cov}(\tilde{\sigma}^2_\beta, \tilde{\sigma}^2_e) \\ \text{cov}(\tilde{\sigma}^2_\alpha, \tilde{\sigma}^2_e) & \text{cov}(\tilde{\sigma}^2_\beta, \tilde{\sigma}^2_e) & \text{var}(\tilde{\sigma}^2_e) \end{bmatrix}$$

$$= 2\underline{T}^{-1} = 2\begin{bmatrix} t_{\alpha\alpha} & t_{\alpha\beta} & t_{\alpha e} \\ t_{\alpha\beta} & t_{\beta\beta} & t_{\beta e} \\ t_{\alpha e} & t_{\beta e} & t_{ee} \end{bmatrix}^{-1} \tag{23}$$

where the elements of $\underline{T}$ are as indicated for (13); for example, $t_{\alpha\beta} = tr(\underline{V}^{-1}\underline{V}_{\alpha}\underline{V}^{-1}\underline{V}_{\beta})$, with $\underline{V}_{\alpha} = \partial\underline{V}/\partial\sigma^2_{\alpha}$ and similarly for $\underline{V}_{\beta}$ and $\underline{V}_{e}$. The algebraic details involved in finding $\underline{V}^{-1}$ from $\underline{V}$ as given in (19)-(22), and then its products with $\underline{V}_{\alpha}$, $\underline{V}_{\beta}$ and $\underline{V}_{e}$ and hence the t's, are indicated in the appendix. The results are as follows.

Define

$$m_{ij} = n_{ij}\,\sigma_\beta^2 + \sigma_e^2 \;, \tag{24}$$

$$A_{ipq} = \sum_{j=1}^{c_i} \frac{n_{ij}^p}{m_{ij}^2} \;, \text{ for integers } p \text{ and } q, \tag{25}$$

and

$$q_i = 1 + \sigma_\alpha^2 A_{i11} \,. \tag{26}$$

Then the elements of $\underline{T}$ are

$$t_{\alpha\alpha} = \sum_{i=1}^{a} A_{i11}^2 / q_i^2 \;, \tag{27}$$

$$t_{\alpha\beta} = \sum_{i=1}^{a} A_{i22}/q_i^2 \;, \tag{28}$$

$$t_{\alpha e} = \sum_{i=1}^{a} A_{i12}/q_i^2 \;, \tag{29}$$

$$t_{\beta\beta} = \sum_{i=1}^{a} \left(A_{i22} - 2\sigma_\alpha^2 A_{i33}/q_i + \sigma_\alpha^4 A_{i22}^2/q_i^2\right) , \tag{30}$$

$$t_{\beta e} = \sum_{i=1}^{a} \left(A_{i12} - 2\sigma_\alpha^2 A_{i23}/q_i + \sigma_\alpha^4 A_{i12}A_{i22}/q_i^2\right) , \tag{31}$$

and

$$t_{ee} = \sum_{i=1}^{a} \left(A_{i02} - 2\sigma_\alpha^2 A_{i13}/q_i + \sigma_\alpha^4 A_{i12}^2/q_i^2\right) + \left(n_{..} - c_{.}\right)/\sigma_e^4 \tag{32}$$

Using these terms as the elements of $\underline{T}$ leads to no explicit expression for the elements of $\underline{V}_{ML} = 2\underline{T}^{-1}$ of (23). Furthermore, the t's involve the unknown variance components; replacing them by estimates of some kind and calculating the corresponding values of the t's and hence $\underline{T}^{-1}$ provides an estimate of $\underline{V}_{ML}$, one that is dependent upon the estimates of the components used. As an estimator of $\underline{V}_{ML}$ it has no known optimal properties; but no alternative estimator suggests itself.

Balanced data

A partial check on the validity of the above results is provided by investigating their behaviour for balanced data. We find, as would be expected, that they yield the variances of the analysis of variance estimators of the components of variance. This is now indicated.

With balanced data, $n_{ij} = n$ for all i and j and $c_i = c$ for all i, so giving

$$m_{ij} = n\beta + e, \quad A_{ipq} = \frac{cn^p}{(n\beta+e)^q} \text{ and } q_i = \frac{cn\alpha + n\beta + e}{n\beta + e}$$

from (24)-(26). Then, on writing

$$x = \frac{a(c-1)}{(n\beta+e)^2}, \quad y = \frac{a}{(cn\alpha+n\beta+e)^2} \text{ and } z = \frac{ac(n-1)}{e^2}$$

$\underline{T}$ becomes, as shown in the appendix,

$$T = \begin{bmatrix} c^2n^2y & cn^2y & cny \\ cn^2y & n^2(x+y) & n(x+y) \\ cny & n(x+y) & x+y+z \end{bmatrix} .$$

To obtain $\underline{T}^{-1}$ we note that its determinant is

$$|T| = c^2n^4xyz$$

and, for example, the

cofactor of the (3,3) term in $|T|$ is $c^2n^4y(x+y-y) = c^2n^4xy$

so that the

(3,3) term in $\underline{T}^{-1}$ is $c^2n^4xy/c^2n^4xyz = 1/z$.

This, from (23), gives

$$\text{var}(\tilde{\sigma}^2_e) = 2/z = \frac{2\sigma^4_e}{ac(n-1)},$$

the familiar result. Similarly, the

cofactor of the (2,2) term in $\underline{T}$ is $c^2n^2y(x+y+z-y) = c^2n^2y(x+z)$

so that the

(2,2) term in $\underline{T}^{-1}$ is $c^2n^2y(x+z)/c^2n^4xyz = (x+z)/n^2xz$.

And from (23) this gives

$$\mathrm{var}(\tilde{\sigma}_\beta^2) = \frac{2(x+z)}{n^2xy} = \frac{2}{n^2}\left(\frac{1}{x} + \frac{1}{z}\right) = \frac{2}{n^2}\left[\frac{(n\sigma_\beta^2 + \sigma_e^2)^2}{a(c-1)} + \frac{\sigma_e^{\ 4}}{ac(n-1)}\right] ,$$

again a familiar result.

<u>The 1-way classification</u>

A further partial check on results (27)-(32) is that they reduce to those of the 1-way classification when the model is suitably amended by either of the two possible ways. One way is to ignore the β's in (16) and put $\sigma_\beta^2 \equiv 0$. The effect of this on the t's is to ignore $t_{\beta\beta}$, $t_{\alpha\beta}$ and $t_{\beta e}$ and, on writing

$$w_i \text{ for } \frac{n_{i.}e}{n_{i.}\alpha+e} ,$$

the others become, as shown in the appendix

$$t_{\alpha\alpha} = e^{-2}\Sigma w_i^2 , \qquad t_{\alpha e} = e^{-2}\Sigma w_i^2/n$$

$$\text{and} \qquad t_{ee} = e^{-2}(\Sigma w_i^2/n_i^2 + n_{..} - a) . \tag{33}$$

Hence the determinant of $\underline{T}$ is

$$|\underline{T}| = t_{\alpha\alpha}t_{ee} - t_{\alpha e}^2$$

$$= e^{-4}\left[\Sigma w_i^2\ \Sigma w_i^2/n_{i.} + (n_{..}-a)\ \Sigma w_i^2 - (\Sigma w_i^2/n_{i.})^2\right] , \tag{34}$$

which, on making use of Lagrange's identity (see Appendix), reduces to

$$|T| = e^{-4}D \text{ where } D = n_{..}\Sigma w_i^2 - (\Sigma w_i)^2 . \tag{35}$$

Hence

$$\mathrm{var}(\tilde{\sigma}_e^2) = 2t_{\alpha\alpha}/D = 2\sigma_e^4\ \Sigma w_i^2/D$$

$$\mathrm{var}(\tilde{\sigma}_\alpha^{\ 2}) = 2t_{ee}/D = 2\sigma_e^4\ (\Sigma w_i^2/n_i^2 + n_{..} - a)/D \tag{36}$$

$$\text{and } \mathrm{cov}(\tilde{\sigma}_\alpha^2\tilde{\sigma}_e^2) = -2t_{\alpha e}/D = (-2\sigma_e^4\ \Sigma w_i^2/n_{i.})/D .$$

as given in Searle (1956).

The other way of reducing the 2-way nested classification model to a 1-way classification is to ignore the α's in (16), treat the double subscript ij as a single subscript r, and put $\sigma_\alpha^2 \equiv 0$. This gives a 1-way classification with $c_.$ classes and n_r observations in the r'th class, r = 1,2, ...c. On defining $w_r = n_r e/(n_r\beta+e)$ the values of $t_{\beta\beta}$, $t_{\beta e}$ and t_{ee} become anologous to those in (33), with results similar to (36).

Estimation of μ

We saw in (14) that the maximum likelihood estimator of the mean in a random model is $\tilde{\mu} = \underline{1}'\tilde{\underline{V}}^{-1}\underline{y}/\underline{1}'\tilde{\underline{V}}^{-1}\underline{1}$ where $\tilde{\underline{V}}^{-1}$ is in terms of the M.L. estimators of the variance components. Since these estimators cannot be obtained explicitly we derive

$$\mu^o = \underline{1}'\underline{V}^{-1}\underline{y}/\underline{1}'\underline{V}^{-1}\underline{1} \;,$$

where the denominator $\underline{1}'\underline{V}^{-1}\underline{1}$ is the variance of $\tilde{\mu}$, as in (15). Then $\tilde{\mu}$ is μ^o with the variance components replaced by their maximum likelihood estimates.

To consider the numerator of μ^o write the vector of observations in the i'th a-class as

$$\underline{y}_i' = (y_{i11}, \ldots, y_{i1n_{i1}}, \ldots \ldots, y_{ij1}, \ldots, y_{ijn_{ij}}, \ldots \ldots, y_{iN_i1}, \ldots, y_{iN_in_{iN_i}}).$$

Then the vector of all observations is

$$\underline{y} = (\underline{y}_1' \; \underline{y}_2' \ldots \underline{y}_a')$$

and

$$\underline{y}'\underline{V}^{-1} = \left(\underline{y}_1'\underline{V}_1^{-1} \quad \underline{y}_2'\underline{V}_2^{-1} \;\ldots\; \underline{y}_a'\underline{V}_a^{-1}\right).$$

Hence the numerator of μ^o is

$$\underline{1}'\underline{V}^{-1}\underline{y} = \sum_{i=1}^{a} (\Sigma \text{ elements in } \underline{y}_i'\underline{V}_i^{-1})$$

and, as shown in the appendix, this reduces to

$$\underline{1}'\underline{V}^{-1}\underline{y} = \sum_{i=1}^{a} \frac{1}{q_i} \sum_{j=1}^{c_i} \frac{n_{ij}}{m_{ij}} \bar{y}_{ij.} \,. \tag{37}$$

Similarly, the denominator of μ^o and variance of $\tilde{\mu}$ is

$$\underline{1}'\underline{V}^{-1}\underline{1} = \Sigma(\text{all elements in } \underline{V}^{-1})$$

$$= \sum_{i=1}^{a} (\Sigma \text{ all elements in } \underline{V}_i^{-1})$$

and this reduces to

$$\underline{1}'\underline{V}^{-1}\underline{1} = \sum_{i=1}^{a} \frac{1}{q_i} \sum_{j=1}^{c_i} \frac{n_{ij}}{m_{ij}} \, . \tag{38}$$

Hence

$$\mu^o = \sum_{i=1}^{a} \frac{1}{q_i} \sum_{j=1}^{c_i} \frac{n_{ij}}{m_{ij}} \bar{y}_{ij.} \Bigg/ \sum_{i=1}^{a} \frac{1}{q_i} \sum_{j=1}^{c_i} \frac{n_{ij}}{m_{ij}}$$

$$= \sum_i \sum_j k_{ij} \bar{y}_{ij.} \Bigg/ \sum_i \sum_j k_{ij} \tag{39}$$

where

$$k_{ij} = \frac{1}{\left(\sigma_\beta^2 + \sigma_e^2/n_{ij}\right)\left(1 + \sigma_\alpha^2 \sum_j \frac{1}{\sigma_\beta^2 + \sigma_e^2/n_{ij}}\right)}$$

In cases where variance components estimates have been obtained, iteratively by the Hartley and Rao (1967) methods for example, replacing σ_α^2, σ_β^2 and σ_e^2 by their corresponding estimates in k_{ij}, and using the resulting values in μ^o leads to μ^o being $\tilde{\mu}$.

As partial check on (39) we can show that when the model is reduced to that of a 1-way classification μ^o reduces to its familiar value. Thus on putting $\sigma_\beta^2 = 0$ we get

$$k_{ij} = \frac{n_{ij}}{\sigma_e^2(1 + \sigma_\alpha^2 \sum_j n_{ij}/\sigma_e^2)} = \frac{n_{ij}}{\sigma_e^2 + n_{i.}\sigma_\alpha^2}$$

and so (39) reduces to

$$\mu^{o} = \sum_{i=1}^{a} \frac{\bar{y}_{i..}}{\sigma_\alpha^2 + \sigma_e^2/n_{i.}} \Big/ \sum_{i=1}^{a} \frac{1}{\sigma_\alpha^2 + \sigma_e^2/n_{i.}} ,$$

the familiar weighted mean of the $\bar{y}_{i..}$'s, weighted inversely by their variances. If, instead, σ_α^2 is put equal to zero μ^{o} becomes

$$\mu^{o} = \sum_i \sum_j \frac{y_{ij.}}{\sigma_\beta^2 + \sigma_e^2/n_{ij}} \Big/ \sum_i \sum_j \frac{1}{\sigma_\beta^2 + \sigma_e^2/n_{ij}} .$$

Acknowledgement

The author is grateful for the support provided for this work by Research Grant GM 13225-04 from the National Institute of General Medical Science, Bethesda, Maryland.

References

Blischke, W. R. (1968). Variances of moment estimators of variance components in the unbalanced r-way classification. Biometrics, 24, 527-540.

Blischke, W. R. (1966). Variances of estimates of variance components in a three-way classification. Biometrics, 22, 553-565.

Crump, S. L. (1951). Present status of variance component analysis. Biometrics, 7, 1-16.

Eisenhart, C. (1947). The assumptions underlying the analysis of variance. Biometrics, 3, 1-21.

Hartley, H. O. (1967). Expectations, variances and covariances of ANOVA mean squares by 'synthesis'. Biometrics, 23, 105-114.

Hartley, H. O. and Rao, J. N. K. (1967). Maximum likelihood estimation for the mixed analysis of variance model. Biometrika, 54, 93-108.

Herbach, L. H. (1959). Properties of Model-II type analysis of variance tests. Ann. Math. Stat., 30, 939-959.

Henderson, C. R. (1953). Estimation of variance and covariance components. Biometrics, 9, 226-252.

Koch, G. G. (1967). A procedure to estimate the population mean in random effects models. Technometrics, 9, 577-586.

Mahamunulu, D. M. (1963). Sampling variances of the estimates of variance components in the unbalanced 3-way nested classification. Ann. Math. Stat., 34, 521-527.

Rao, J. N. K. (1968). On expectations, variances and covariances of ANOVA mean squares by 'synthesis'. Biometrics, 24, 963-978.

Scheffe, H. (1959). "The Analysis of Variance". Wiley, New York.

Searle, S. R. (1956). Matrix methods in variance and covariance components analysis. Ann. Math. Stat., 27, 737-748.

Searle, S. R. (1958). Sampling variances of estimates of components of variance. Ann. Math. Stat., 29, 167-178.

Searle, S. R. (1961). Variance components in the unbalanced 2-way nested classification. Ann. Math. Stat., 32, 1161-1166.

Searle, S. R. (1968). Another look at Henderson's methods of estimating variance components (with discussion). Biometrics, 24, 749-788.

Thompson, W. A. Jr. (1961). Negative estimates of variance components: an introduction. Bulletin, International Institute of Statistics, 34, 1-4.

Urquhart, N. S. (1962). The repeated design and further considerations of the general two-way design. M.S. Thesis, Colorado State University, Fort Collins, Colorado.

Appendix

Inverse of V

Since $\underset{\sim}{V} = \sum_{i=1}^{a+} \underset{\sim}{V}_i$, as in (19) it is clear from the nature of a direct sum that

$$\underset{\sim}{V}^{-1} = \sum_{i=1}^{a+} \underset{\sim}{V}_i^{-1} .$$

For this, $\underset{\sim}{V}_i^{-1}$ is derived from the following theorem taken from Urquhart (1962).

Theorem Consider a matrix $\underset{\sim}{A}$ partitioned as

$$\underset{\sim}{A} = \left\{\underset{\sim}{A}_{pq} \text{ of order } n_p \times n_q\right\} \text{ for } p, q = 1, 2, \ldots, N \qquad \text{(A1)*}$$

such that

$$\underset{\sim}{A}_{pp} = b_p \underset{\sim}{I}_{n_p} + g_{pp} \underset{\sim}{J}_{n_p} \qquad \text{(A2)}$$

and $$\underset{\sim}{A}_{pq} = g_{pq} \underset{\sim}{J}_{n_p \times n_q} \quad \text{for } p \neq q \qquad \text{(A3)}$$

with $$\underset{\sim}{G} = \left\{g_{pq}\right\} . \qquad \text{(A4)}$$

Then the inverse of $\underset{\sim}{A}$ is

$$\underset{\sim}{A}^{-1} = \left\{(\underset{\sim}{A}^{-1})_{pq} \text{ of order } n_p \times n_q\right\} . \qquad \text{(A5)}$$

with $$(\underset{\sim}{A}^{-1})_{pp} = (1/b_p)\underset{\sim}{I}_{n_p} + k_{pp}\underset{\sim}{J}_{n_p} \qquad \text{(A6)}$$

and $$(\underset{\sim}{A}^{-1})_{pq} = k_{pq}\underset{\sim}{J}_{n_p \times n_q} \quad \text{for } p \neq q \qquad \text{(A7)}$$

where $$\underset{\sim}{K} = \left\{k_{pq}\right\}$$

$$= \left[(\underset{\sim}{G}\underset{\sim}{D} + \underset{\sim}{B})^{-1} - \underset{\sim}{B}^{-1}\right] \underset{\sim}{D}^{-1} \qquad \text{(A8)}$$

with $$\underset{\sim}{D} = \text{diag}\left\{n_1, \ldots, n_N\right\} \text{ and } \underset{\sim}{B} = \text{diag}\left\{b_1, \ldots, b_N\right\} . \qquad \text{(A9)}$$

The notation in (A9) indicates that $\underline{D}$ and $\underline{B}$ are diagonal matrices, and in (A5) - (A7) the notation $(\underline{A}^{-1})_{pq}$ does not indicate the inverse of a matrix; it is the pq'th sub-matrix of the inverse of $\underline{A}$.

Comparing the definition of $\underline{V}_i$ in (20) - (22) with that of $\underline{A}$ in (A1) - (A3) indicates that in applying the theorem to find $\underline{V}_i^{-1}$ the N of the theorem is c_i, b_p is e, g_{pp} is $\alpha + \beta$ and g_{pq} is α for $p \neq q$, for $p, q = 1, 2, \ldots, c_i$. Hence $\underline{G}$ of (A4), which we now subscript with i to go with $\underline{V}_i$, is

$$\underline{G}_i = \beta \underline{I}_{c_i} + \alpha \underline{J}_{c_i} \; . \tag{A10}$$

Then from (A5)

$$\underline{V}_i^{-1} = \left\{ (\underline{V}_i^{-1})_{jj'} \text{ of order } n_{ij} \times n_{ij'} \right\} \text{ for } j, j' = 1, 2, \ldots, c_i \tag{A11}$$

with, from (A6) and (A7),

$$(\underline{V}_i^{-1})_{jj} = (1/e)\, \underline{I}_{n_{ij}} + h_{i,jj} \underline{J}_{n_{ij}} \tag{A12}$$

and

$$(\underline{V}_i^{-1})_{jj'} = h_{i,jj'} \underline{J}_{n_{ij} \times n_{ij'}} \quad \text{for } j \neq j' \tag{A13}$$

where, from (A8)

$$\left\{ h_{i,jj'} \right\} \text{ for } j,j' = 1, 2, \ldots, c_i, = \underline{H}_i = \left[(\underline{G}_i \underline{D}_i + \underline{B}_i)^{-1} - \underline{B}_i^{-1} \right] \underline{D}_i^{-1} \tag{A14}$$

with (A9) giving

$$\underline{D}_i = \text{diag} \left\{ n_{i1}, \ldots, n_{ic_i} \right\} \text{ and } \underline{B}_i = \underline{e}\underline{I}_{c_i} \; .$$

Hence to obtain $\underline{V}_i^{-1}$ we need $\underline{H}_i$ of (A14), first finding the inverse of

$$\underline{G}_i\underline{D}_i + \underline{B}_i = \begin{bmatrix} n_{i1}(\alpha+\beta)+e & n_{i2}\alpha & \cdots & n_{ic_i}\alpha \\ n_{i1}\alpha & n_{i2}(\alpha+\beta)+e & \cdots & n_{ic_1}\alpha \\ \vdots & \vdots & & \\ n_{i1}\alpha & n_{i2}\alpha & \cdots & n_{ic_i}(\alpha+\beta)+e \end{bmatrix}. \quad (A15)$$

For convenience define,

$$m_{ij} = n_{ij}\beta + e, \quad (A16)$$

$$p_i = \prod_{j=1}^{c_i} m_{ij}, \quad (A17)$$

and

$$q_i = 1 + \alpha \sum_{j=1}^{c_i} \frac{n_{ij}}{m_{ij}}, \quad (A18)$$

where m_{ij} and q_i are exactly as in (24) and (25). Then

$$\underline{G}_i\underline{D}_i + \underline{B}_i = \begin{bmatrix} n_{i1}\alpha + m_{i1} & n_{i2}\alpha & \cdots & n_{ic_i}\alpha \\ n_{i1}\alpha & n_{i2}\alpha + m_{i2} & & n_{ic_i}\alpha \\ \vdots & \vdots & & \vdots \\ n_{i1}\alpha & n_{i2}\alpha & \cdots & n_{ic_i} + m_{ic_i} \end{bmatrix}$$

with the determinant being, from diagonal expansion,

$$|\underline{G}_i\underline{D}_i + \underline{B}_i| = \prod_{j=1}^{c_i} m_{ij}\left(1 + \sum_{j=1}^{c_i} \frac{n_{ij}\alpha}{m_{ij}}\right) = p_i q_i \ . \quad (A20)$$

To find the inverse of $\underline{G}_i\underline{D}_i + \underline{B}_i$ we find the cofactors of its elements. That of its j'th diagonal element is, by analogy with (A20)

$$(1/m_{ij}) \prod_{j=1}^{c_i} m_{ij}\left(1 + \sum_{j=1}^{c_i} \frac{n_{ij}\alpha}{m_{ij}} - \frac{n_{ij}\alpha}{m_{ij}}\right) = \frac{p_i}{m_{ij}}\left(q_i - \frac{n_{ij}\alpha}{m_{ij}}\right) ; \quad (A21)$$

and that of its (jj')'th off-diagonal element for $j \neq j'$ is $(-1)^{j+j'}|\underline{M}_{i,jj'}|$ where $\underline{M}_{i,jj'}$ is the corresponding minor. In subtracting the $(j' - 1)$'th row of $|\underline{M}_{i,jj'}|$ - which, for $j < j'$, has come from the j' 'th row of $\underline{G}_i\underline{D}_i + \underline{B}_i$ - from every other row of $|\underline{M}_{i,jj'}|$ we find that for $t \neq j \neq j'$ all elements $n_{it}(\alpha + \beta) + e$ become $n_{it}\beta + e$ and elements $n_{it}\alpha$ become zero; and the only non-zero element in the j'th column is $n_{ij}\alpha$ in the $(j' - 1)$'th position. Expanding $|\underline{M}_{i,jj'}|$ by elements of this column gives

$$|\underline{M}_{i,jj'}| = (-1)^{j'-1+j} n_{ij}\alpha \prod_{t\neq j\neq j'}^{c_i} (n_{ij}\beta + e) = (-1)^{j'-1+j} n_{ij}\alpha p_i/m_{ij}m_{ij'} .$$

This is for $j < j'$. When $j > j'$, the effect is to interchange j and j' in the above result, which merely replaces n_{ij} by $n_{ij'}$. Hence the cofactor of the element in the (jj')'th position, for $j \neq j'$, is

$$(-1)^{j+j'}|\underline{M}_{i,jj'}| = -n_{ij}\alpha p_i/m_{ij}m_{ij'} \quad . \tag{A22}$$

Dividing (A21) and (A22) by (A20) shows that $(\underline{G}_i\underline{D}_i + \underline{B}_i)^{-1}$ has its

$$\text{j'th diagonal element} = \frac{1}{m_{ij}} - \frac{n_{ij}\alpha}{m_{ij}^2\, q_i}$$

and its

$$\text{(jj')'th element, } j \neq j', = \frac{-\, n_{ij'}\alpha}{m_{ij}\, m_{ij'}\, q_i} .$$

Therefore, with $\underline{D}_i^{-1} = \text{diag}\left\{1/n_{i1}, \ldots, 1/n_{ic_i}\right\}$ and $\underline{B}_i^{-1} = (1/e)\underline{I}_{c_i}$, the matrix $\underline{H}_i$ of (A14) has diagonal elements

$$h_{i,jj} = \left[\frac{1}{m_{ij}} - \frac{n_{ij}\,\alpha}{m_{ij}^2\, q_i} - \frac{1}{e}\right]\frac{1}{n_{ij}} \tag{A23}$$

which, from (A16) reduces to

$$h_{i,jj} = \frac{-\beta}{m_{ij}e} - \frac{\alpha}{m_{ij}^2\, q_i} \quad \text{for } j = 1, 2, \ldots, c_i; \tag{A24}$$

and off-diagonal elements

$$h_{i,jj'} = \frac{-\alpha}{m_{ij}\, m_{ij'}\, q_i} \quad \text{for } j \neq j' = 1, 2, \ldots, c_i . \tag{A25}$$

With these terms used in (A12) and (A13). $\underset{-}{V}_i^{-1}$ of (A11) is determined and so $\underset{-}{V}^{-1} = \sum_{i=1}^{a}{}^{+} \underset{-}{V}_i$ is known.

Elements of T

To derive $t_{\alpha\beta} = tr(\underset{-}{V}^{-1}\underset{-}{V}_\alpha\underset{-}{V}^{-1}\underset{-}{V}_\beta)$ for example, we need the differentials of $\underset{-}{V}$ with respect to α and β. This, because $\underset{-}{V} = \sum_{i=1}^{a}{}^{+} \underset{-}{V}_i$, requires the differentials of $\underset{-}{V}_i$, and from the definitions given in (20) - (22) it is readily seen that

$$\underset{-}{V}_{i,\alpha} = \partial\underset{-}{V}_i/\partial\sigma_\alpha^2 = \underset{-}{J}_{n_{i.}} ,$$

$$\underset{-}{V}_{i,\beta} = \partial\underset{-}{V}_i/\partial\sigma_\beta^2 = \sum_{j=1}^{c_i}{}^{+} \underset{-}{J}_{n_{ij}} ,$$

and

$$\underset{-}{V}_{i,e} = \partial\underset{-}{V}_i/\partial\sigma_e^2 = \underset{-}{I}_{n_{i.}} .$$

With these values, and obtaining $\underset{-}{V}_i^{-1}$ of (A11) by using (A23) - (A25) in (A12) and (A13) we now derive the t's. First, $t_{\alpha\alpha}$, utilizing $\underset{-}{V} = \sum_{i=1}^{a}{}^{+} \underset{-}{V}_i$ and $\underset{-}{V}^{-1} = \sum_{i=1}^{a}{}^{+} \underset{-}{V}_i^{-1}$:

$$t_{\alpha\alpha} = tr(\underset{-}{V}^{-1}\underset{-}{V}_\alpha\underset{-}{V}^{-1}\underset{-}{V}_\alpha)$$

$$= \sum_{i=1}^{a} tr(\underset{-}{V}_i^{-1}\underset{-}{V}_{i,\alpha}\underset{-}{V}_i^{-1}\underset{-}{V}_{i,\alpha})$$

$$= \sum_{i=1}^{a} tr(\underset{-}{V}_i\underset{-}{J}_{n_{i.}})^2$$

$$= \sum_{i=1}^{a} \sum_{r=1}^{n_{i.}} \text{i.p.o. (r'th row of } \underset{-}{V}_i^{-1}\underset{-}{J}_{n_{i.}})$$

$$\text{and (r'th column of } \underset{-}{V}_i^{-1}\underset{-}{J}_{n_{i.}})$$

where i.p.o. stands for "inner product of". Now in $\underline{V}_i^{-1}\underline{J}_{n_i}$ every element in a row is the same; and so the columns are equal. Therefore

$$t_{\alpha\alpha} = \sum_{i=1}^{a} \sum_{r=1}^{n_{i.}} (\text{element of r'th row of } \underline{V}_i^{-1}\underline{J}_{n_{i.}})(\text{column sum of } \underline{V}_i^{-1}\underline{J}_{n_{i.}}) \quad (A26)$$

Now, by the nature of $\underline{V}_i^{-1}$, the $n_{i.}$ rows of $\underline{V}_i^{-1}\underline{J}_{n_{i.}}$ are grouped naturally into c_i sets of n_{ij} rows each, for $j = 1, 2, \ldots, c_i$. Therefore instead of considering the r'th row of $\underline{V}_i^{-1}\underline{J}_{n_{i.}}$ in $t_{\alpha\alpha}$ we consider the k'th row in the j'th set of rows, and replace the summation $\sum_{r=1}^{n_i}$ by $\sum_{j=1}^{c_i}\sum_{k=1}^{n_{ij}}$. Then

an element in the k'th row of the j'th set of rows of $\underline{V}_i^{-1}\underline{J}_{n_{i.}}$

$= \Sigma$ elements in the k'th row of the j'th set of rows of $\underline{V}_i^{-1}$,

and from the nature of (A12) and (A13) this is

$$= (1/e + n_{ij}\, h_{i,jj}) + \sum_{j'\neq j} n_{ij'}\, h_{i,jj'}$$

which, on substituting from (A23) and (A25) is

$$= \frac{1}{m_{ij}} - \frac{n_{ij}\,\alpha}{m_{ij}^2\, q_i} - \sum_{j'\neq j} \frac{n_{ij'}\,\alpha}{m_{ij}\, m_{ij'}\, q_i}$$

$$= \frac{1}{m_{ij}} - \frac{n_{ij}\,\alpha}{m_{ij}^2\, q_i} - \frac{\alpha}{m_{ij}\, q_i}\left(\sum_{j'=1}^{c_i} \frac{n_{ij'}}{m_{ij'}} - \frac{n_{ij}}{m_{ij}}\right)$$

$$= \frac{1}{m_{ij}} - \frac{\alpha}{m_{ij}\, q_i}\sum_{j=1}^{c_i}\frac{n_{ij}}{m_{ij}} = \frac{1}{m_{ij}} - \frac{1}{m_{ij}\, q_i}(q_i - 1)$$

$$= \frac{1}{m_{ij}\, q_i}, \quad (A27)$$

on making use of (A16) and (A18) in this reduction. Thus an element in the k'th row of the j'th set of rows of $\underline{V}_i^{-1}\underline{J}_{n_{i.}}$ is $1/m_{ij}q_i$, and summing this over all rows gives

$$\text{column sum of } \underset{\sim}{V}_i^{-1}\underset{\sim}{J}_{n_{i.}} = \sum_{j=1}^{c_i}\sum_{k=1}^{n_{ij}} \frac{1}{m_{ij}\, q_i} = \frac{1}{q_i}\sum_{j=1}^{c_i}\frac{n_{ij}}{m_{ij}} . \tag{A28}$$

Substituting (A27) and (A28) in (A26) gives

$$t_{\alpha\alpha} = \sum_{i=1}^{a}\sum_{j=1}^{c_i}\sum_{k=1}^{n_{ij}} \frac{1}{m_{ij}\, q_i}\left(\frac{1}{q_i}\sum_{j=1}^{c_i}\frac{n_{ij}}{m_{ij}}\right) = \sum_{i=1}^{a}\frac{1}{q_i^2}\left(\sum_{j=1}^{c_i}\frac{n_{ij}}{m_{ij}}\right)^2 .$$

Now define

$$A_{ipq} = \sum_{j=1}^{c_i}\frac{n_{ij}^p}{m_{ij}^q} \qquad \text{for integers p and q,}$$

as in (25), noting from (A18) that

$$q_i = 1 + \alpha A_{i11} = 1 + \sigma_\alpha^2 A_{i11}$$

as in (26). Then $t_{\alpha\alpha}$ gets written as

$$t_{\alpha\alpha} = \sum_{i=1}^{a} A_{i11}^2/q_i^2$$

as shown in (27).

The definitions and procedures introduced in deriving this result are used repeatedly below in obtaining the other t's.

$$t_{\alpha\beta} = \mathrm{tr}(\underset{\sim}{V}^{-1}\underset{\sim}{V}_\alpha \underset{\sim}{V}^{-1}\underset{\sim}{V}_\beta)$$

$$= \sum_{i=1}^{a}\mathrm{tr}(\underset{\sim}{V}_i^{-1}\underset{\sim}{V}_{i,\alpha}\underset{\sim}{V}_i^{-1}\underset{\sim}{V}_{i,\beta})$$

$$= \sum_i \mathrm{tr}(\underset{\sim}{V}_i^{-1}\ \underset{\sim}{J}_{n_{i.}}\ \underset{\sim}{V}_i^{-1}\sum_j{}^{+}\ \underset{\sim}{J}_{n_{ij}})$$

$$= \sum_i\sum \text{ i.p.o. (r'th row of } \underset{\sim}{V}_i^{-1}\underset{\sim}{J}_{n_{i.}}) \text{ and (r'th column of } \underset{\sim}{V}_i^{-1}\sum_j{}^{+}\ \underset{\sim}{J}_{n_{ij}})$$

$$= \sum_i\sum_{r=1}^{n_{i.}} \text{(element of r'th row of } \underset{\sim}{V}_i^{-1}\underset{\sim}{J}_{n_{i.}})(\Sigma \text{ elements in r'th column of } \underset{\sim}{V}_i^{-1}\sum_j{}^{+}\ \underset{\sim}{J}_{n_{ij}})$$

$$= \sum_{i}^{a} \sum_{j}^{c_i} \sum_{k=1}^{n_{ij}} (1/m_{ij}\ q_i)(\Sigma \text{ elements in k'th column of j'th set of columns of } \underset{-}{V}_i^{-1} \Sigma_j^{+} \underset{-}{J}_{n_{ij}})$$

$$= \sum_i \sum_j \sum_k (1/m_{ij}\ q_i) \left\{\Sigma \text{ elements in the column } [(0 \ldots 0\ \underset{-}{1}_{n_{ij}}\ 0 \ldots 0)\underset{-}{V}_i^{-1}]'\right\}$$

$$= \sum_i \sum_j \sum_k (1/m_{ij}\ q_i)(\Sigma \text{ all elements in the j'th set of columns of } \underset{-}{V}_i^{-1})$$

$$= \sum_i \sum_j \sum_k (1/m_{ij}\ q_i) \left\{\Sigma \text{ all elements in } [(\underset{-}{V}_i^{-1})_{jj} + \sum_{j' \neq j} (\underset{-}{V}_i^{-1})_{jj'}]\right\}$$

$$= \sum_i \sum_j \sum_k (1/m_{ij}\ q_i)(n_{ij}/e + n_{ij}^2\ h_{i,jj} + \sum_{j' \neq j} n_{ij}\ n_{ij'}\ h_{i,jj'})$$

$$= \sum_i \sum_j \sum_k (1/m_{ij}\ q_i)(n_{ij}/m_{ij}\ q_i) \text{ from (A27)}$$

$$= \sum_i \frac{1}{q_i^2} \sum_j \frac{n_{ij}^2}{m_{ij}^2}$$

$$= \sum_i A_{i22}/q_i^2 \text{ as shown in (28).}$$

Similarly

$$t_{\alpha e} = tr(\underset{-}{V}^{-1}\underset{-}{V}_{\alpha}\underset{-}{V}^{-1}\underset{-}{V}_e)$$

$$= \sum_{i=1}^{a} tr(\underset{-}{V}_i^{-1}\underset{-}{V}_{i,\alpha}\underset{-}{V}_i^{-1}\underset{-}{V}_{i,e})$$

$$= \sum_i tr(\underset{-}{V}_i^{-1}\ \underset{-}{J}_{n_{i.}}\ \underset{-}{V}_i^{-1})$$

$$= \sum_i \sum_r (\text{element in r'th row of } \underset{-}{V}_i^{-1}\underset{-}{J}_{n_{i.}})(\Sigma \text{ elements in r'th column of } \underset{-}{V}_i^{-1})$$

$$= \sum_i \sum_j \sum_k (1/m_{ij}\ q_i)(1/e + n_{ij}\ h_{i,jj} + \sum_{j' \neq j} n_{ij'}\ h_{i,jj'})$$

$$= \sum_i \sum_j \frac{n_{ij}}{m_{ij}\ q_i} \left(\frac{1}{m_{ij}\ q_i}\right) \text{ from (A26)}$$

$$= \sum_i A_{i12}/q_i^2 \text{ , given in (29).}$$

In deriving the term

$$t_{\beta\beta} = \mathrm{tr}(\underline{V}^{-1}\underline{V}_{\beta}\underline{V}^{-1}\underline{V}_{\beta})$$

$$= \sum_{i} \mathrm{tr}[(\Sigma_{j}^{+}\underline{J}_{n_{ij}})\underline{V}_{i}^{-1}]^{2}$$

we use the partitioning

$$(\Sigma_{j}^{+}\underline{J}_{n_{ij}})\underline{V}_{i}^{-1} = \left\{\underline{P}_{i,jj'}\right\} \text{ for } j,j' = 1,\ 2,\ \ldots,\ c_{i}$$

with $\quad \underline{P}_{i,jj} = (\Sigma_{j}^{+}\underline{J}_{n_{ij}})(\underline{V}_{i}^{-1})_{jj} = (1/e)\underline{J}_{n_{ij}} + n_{ij}\, h_{i,jj}\, \underline{J}_{n_{ij}}$

and $\quad \underline{P}_{i,jj'} = (\Sigma_{j}^{+}\underline{J}_{n_{ij}})(\underline{V}_{i}^{-1})_{jj} = n_{ij}\, h_{i,jj'}\, \underline{J}_{n_{ij} \times n_{ij'}}$.

This gives

$$t_{\beta\beta} = \sum_{i} \mathrm{tr}(\left\{\underline{P}_{i,jj'}\right\} j,j' = 1,\ 2,\ \ldots,\ c_{i})^{2}$$

$$= \sum_{i} \mathrm{tr}(\sum_{j} \underline{P}_{i,jj}^{2} + \sum_{j}\sum_{j'\neq j}\underline{P}_{i,jj'}\underline{P}_{i,j'j})$$

$$= \sum_{i}\sum_{j} \mathrm{tr}(\underline{P}_{i,jj}^{2} + \sum_{j'\neq j}\underline{P}_{i,jj'}\underline{P}_{i,j'j})$$

$$= \sum_{i}\sum_{j} \mathrm{tr}[(1/e^{2})n_{ij}\underline{J}_{n_{ij}} + n_{ij}^{2}h_{i,jj}^{2}n_{ij}\underline{J}_{n_{ij}} + 2(1/e)n_{ij}h_{i,jj}n_{ij}\underline{J}_{n_{ij}}$$

$$+ \sum_{j'\neq j} n_{ij}h_{i,jj'}\underline{J}_{n_{ij} \times n_{ij'}}n_{ij'}h_{i,j'j}\underline{J}_{n_{ij'} \times n_{ij}}]$$

$$= \sum_{i}\sum_{j} [n_{ij}^{2}(1/e + n_{ij}h_{i,jj})^{2} + n_{ij}^{2}\sum_{j'\neq j} n_{ij'}^{2}h_{i,jj'}^{2}]$$

$$= \sum_{i}\sum_{j} \left[n_{ij}^{2}\left(\frac{1}{m_{ij}} - \frac{n_{ij}\,\alpha}{m_{ij}^{2}\,q_{i}}\right)^{2} + n_{ij}^{2}\sum_{j'\neq j}\frac{n_{ij'}\,\alpha^{2}}{m_{ij}^{2}\,m_{ij'}^{2}\,q_{i}^{2}}\right]$$

$$= \sum_{i}\sum_{j} \left[\frac{n_{ij}^{2}}{m_{ij}^{2}}\left(1 + \frac{n_{ij}^{2}\,\alpha^{2}}{m_{ij}^{2}\,q_{i}^{2}} - \frac{2n_{ij}\,\alpha}{m_{ij}\,q_{i}}\right) + \frac{n_{ij}^{2}\,\alpha^{2}}{m_{ij}^{2}\,q_{i}^{2}}\left(\sum_{j'}\frac{n_{ij'}^{2}}{m_{ij'}^{2}} - \frac{n_{ij}^{2}}{m_{ij}^{2}}\right)\right]$$

$$= \sum_i \sum_j \left(\frac{n_{ij}^2}{m_{ij}^2} - \frac{2\alpha\, n_{ij}^3}{m_{ij}^3\, q_i} + \frac{\alpha^2\, n_{ij}^2}{q_i^2\, m_{ij}^2} \sum_{j'} \frac{n_{ij'}^2}{m_{ij'}^2} \right)$$

$$= \sum_{i=1}^{a} (A_{i22} - 2\sigma_\alpha^2\, A_{i33}/q_i + \sigma_\alpha^4 A_{i22}/q_i^2) \text{ as in (30).}$$

The penultimate term is

$$t_{\beta e} = \mathrm{tr}(\underline{V}^{-1}\underline{V}_\beta \underline{V}^{-1}\underline{V}_e)$$

$$= \sum_i \mathrm{tr}(\underline{V}_i^{-1}\underline{V}_{i,\beta}\underline{V}_i^{-1})$$

$$= \sum_i \mathrm{tr}[(\sum_j{}^{+}\underline{J}_{n_{ij}})\underline{V}_i^{-1}\underline{V}_i^{-1}]$$

$$= \sum_i \mathrm{tr}[\sum_j \underline{P}_{i,jj}(\underline{V}_i^{-1})_{jj} + \sum_j \sum_{j'\neq j} \underline{P}_{i,jj'}(\underline{V}_i^{-1})_{j'j}]$$

$$= \sum_i \sum_j \mathrm{tr}\Big\{[(1/e)\underline{J}_{n_{ij}} + n_{ij}h_{i,jj}\underline{J}_{n_{ij}}][(1/e)\underline{I}_{n_{ij}} + h_{i,jj}\underline{J}_{n_{ij}}]$$

$$+ \sum_{j'\neq j} n_{ij}h_{i,jj'}\underline{J}_{n_{ij} \times n_{ij'}} h_{i,jj'}\underline{J}_{n_{ij'} \times n_{ij}}\Big\}$$

$$= \sum_i \sum_j \mathrm{tr}[(1/e^2)\underline{J}_{n_{ij}} + 2(1/e)n_{ij}h_{i,jj}\underline{J}_{n_{ij}} + n_{ij}^2 h_{i,jj}^2 \underline{J}_{n_{ij}}$$

$$+ \sum_{j'\neq j} n_{ij}n_{ij'}h_{i,jj'}^2 \underline{J}_{n_{ij}}]$$

$$= \sum_i \sum_j (n_{ij}/e^2 + 2n_{ij}^2 h_{i,jj}/e + n_{ij}^3 h_{i,jj}^2 + n_{ij}^2 \sum_{j'\neq j} n_{ij'}h_{i,jj'}^2)$$

$$= \sum_i \sum_j [n_{ij}(1/e + n_{ij}h_{i,jj})^2 + n_{ij}^2 \sum_{j'\neq j} n_{ij'}h_{i,jj'}^2]$$

$$= \sum_i \sum_j \left[\frac{n_{ij}}{m_{ij}^2}\left(1 + \frac{n_{ij}^2\alpha^2}{m_{ij}^2 q_i^2} - \frac{2n_{ij}\alpha}{m_{ij}q_i}\right) + \frac{n_{ij}^2\alpha^2}{m_{ij}^2 q_i^2}\left(\sum_{j'} \frac{n_{ij'}}{m_{ij'}^2} - \frac{n_{ij}}{m_{ij}^2}\right)\right]$$

and, similar to the final reduction of $t_{\beta\beta}$, this becomes, as shown in (31),

$$t_{\beta e} = \sum_{i=1}^{a} (A_{i12} - 2\sigma_\alpha^2 A_{i23}/q_i + \sigma_\alpha^4 A_{i12}A_{i22}/q_i^2).$$

Finally we have

$$t_{ee} = \operatorname{tr}(\underset{\sim}{V}^{-1}\underset{\sim}{V}_e\underset{\sim}{V}^{-1}\underset{\sim}{V}_e)$$

$$= \sum_i \operatorname{tr}(\underset{\sim}{V}_i^{-1})^2$$

$$= \sum_i \sum_j \operatorname{tr}\left\{[(\underset{\sim}{V}_i^{-1})_{jj}]^2 + \sum_{j'\neq j}(\underset{\sim}{V}_i^{-1})_{jj'}(\underset{\sim}{V}_i^{-1})_{j'j}\right\}$$

and on substituting from (A12) and (A13) this becomes

$$t_{ee} = \sum_i \sum_j \operatorname{tr}\left\{(1/e^2)\underset{\sim}{I}_{n_{ij}} + 2(1/e)h_{ijj}\underset{\sim}{J}_{n_{ij}} + h^2_{i,jj}n_{ij}\underset{\sim}{J}_{n_{ij}}\right.$$

$$\left. + \sum_{j'\neq j} h_{i,jj'}\underset{\sim}{J}_{n_{ij}} \times n_{ij'}h_{i,jj'}\underset{\sim}{J}_{n_{ij'}} \times n_{ij}\right\}$$

$$= \sum_i \sum_j (n_{ij}/e^2 + 2n_{ij}h_{i,jj}/e + n^2_{ij}h^2_{i,jj} + n_{ij}\sum_{j'\neq j} n_{ij'}h^2_{ijj'})$$

$$= \sum_i \sum_j [(n_{ij} - 1)/e^2 + (1/e + n_{ij}h_{i,jj})^2 + n_{ij}\sum_{j'\neq j} n_{ij'}h^2_{ijj'}]$$

$$= \frac{n_{..} - c_.}{e^2} + \sum_i \sum_j \left[\frac{1}{m^2_{ij}}\left(1 + \frac{n^2_{ij}\alpha^2}{m^2_{ij}q^2_i} - \frac{2n_{ij}\alpha}{m_{ij}q_i}\right) + \right.$$

$$\left. + \frac{n_{ij}}{m^2_{ij}}\frac{\alpha^2}{q^2_i}\left(\sum_{j'\neq j}\frac{n_{ij'}}{m^2_{ij'}} - \frac{n_{ij}}{m^2_{ij}}\right)\right]$$

$$= (n_{..} - c_.)/e^2 + \sum_{i=1}^{a}(A_{i02} - 2\sigma^2_\alpha A_{i13}/q_i + \sigma^4_\alpha A^2_{i12}/q^2_i)$$

as shown in (32).

Balanced Data

With $n_{ij} = n$, $c_i = c$, for all i and j,

$$m_{ij} = n\beta + e \qquad A_{ipq} = \frac{cn^p}{(n\beta + e)^q} \quad \text{and} \quad q_i = \frac{cn\alpha + n\beta + e}{n\beta + e}\,.$$

$$x = \frac{a(c-1)}{(n\beta + e)^2}, \quad y = \frac{a}{(cn\alpha + n\beta + e)^2} \quad \text{and} \quad z = \frac{ac(n-1)}{e^2}$$

we obtain, from (27) - (32):

$$t_{\alpha\alpha} = a\left(\frac{cn}{n\beta + e}\right)^2 \left(\frac{n\beta + e}{cn\alpha + n\beta + e}\right)^2 = c^2n^2y\ ,$$

$$t_{\alpha\beta} = \frac{acn^2}{(n\beta + e)^2}\left(\frac{n\beta + e}{cn\alpha + n\beta + e}\right)^2 = cn^2y\ ,$$

$$t_{\alpha e} = \frac{acn}{(n\beta + e)^2}\left(\frac{n\beta + e}{cn\alpha + n\beta + e}\right)^2 = cny\ ,$$

and

$$t_{\beta\beta} = \frac{acn^2}{(n\beta + e)^2}\left[1 - \frac{2\alpha n}{cn\alpha + n\beta + e} + \frac{c\alpha^2 n^2}{(cn\alpha + n\beta + e)^2}\right]$$

$$= \frac{an^2}{(n\beta + e)^2}\left[\frac{c(cn\alpha + n\beta + e)^2 - 2\ cn\alpha(cn\alpha + n\beta + e)^2 + c^2n^2\alpha^2}{(cn\alpha + n\beta + e)^2}\right]$$

in which the numerator inside the square brackets can be simplified as

$$(n\beta + e)^2 + (c-1)(n\beta + e)^2 + c^3n^2\alpha^2 + 2c^2n\alpha(n\beta + e) - 2cn\alpha(cn\alpha + n\beta + e) + cn^2\alpha^2$$

$$= (n\beta + e)^2 + (c-1)(n\beta + e)^2 + (c-1)c^2n^2\alpha^2 + 2cn\alpha[c(n\beta + e) - (cn\alpha + n\beta + e) + cn\alpha]$$

$$= (n\beta + e)^2 + (c-1)(cn\alpha + n\beta + e)^2$$

and so

$$t_{\beta\beta} = \frac{an^2(c-1)}{(n\beta + e)^2} + \frac{an^2}{(cn\alpha + n\beta + e)^2} = n^2(x + y)\ .$$

Furthermore, since in (31) the powers of n in the A-terms are one less than those in (30)

$$t_{\beta e} = t_{\beta\beta}/n = n(x + y)\ .$$

And similarly for the first term of (32), so that

$$t_{ee} = t_{\beta e}/n + (acn - ac)/e^2 = x + y + z.$$

<u>The 1-way classification</u>

With $\sigma_\beta^2 = 0$ and using $w_i = n_i e/(e + n_i\alpha)$, (25) and (26) give

$$A_{ipq} = e^{-q} \sum_j n_{ij}^p$$

and $$q_i = (e + n_{i.}\alpha)/e = n_{i.}/w_i \ .$$

Thus from (27), (29) and (32)

$$t_{\alpha\alpha} = \Sigma\, e^{-2} n_i^2 w_i^2/n_i^2 = e^{-2}\Sigma\, w_i^2$$

$$t_{\alpha e} = \Sigma\, e^{-2} n_{i.} w_i^2/n_{i.}^2 = e^{-2}\Sigma\, w_i^2/n_{i.}$$

and

$$t_{ee} = \Sigma\left[\frac{c_i}{e^2} - \frac{2\alpha\, e}{e + n_{i.}\alpha}\,\frac{n_{i.}}{e^3} + \frac{e^2 n_{i.}^2}{(e + n_{i.}\alpha)^2 e^4}\right] + \frac{n_{..} - c_.}{e^2}$$

$$= e^{-2}\left[\Sigma(c_i - 1) + \Sigma\left(1 - \frac{\alpha n_{i.}}{e + n_{i.}\alpha}\right)^2\right] + e^{-2}(n_{..} - c_.)$$

$$= e^{-2}(\Sigma\, w_i^2/n_{i.}^2 + n_{..} - a),$$

as shown in (33).

The Lagrange identity is

$$\sum_i a_i^2 \sum_i b_i^2 - \left(\sum_i a_i b_i\right)^2 = \tfrac{1}{2}\sum_{i \neq i'}\sum (a_i b_{i'} - a_{i'} b_i)^2.$$

Hence, with $$w_i = n_{i.}e/(n_{i.}\alpha + e)$$

$$\sum_i w_i^2 \sum_i w_i^2/n_{i.}^2 - \left(\sum_i w_i^2/n_{i.}\right)^2$$

$$= \tfrac{1}{2}\sum_{i\neq i'}\sum (w_i w_{i'}/n_{i'.} - w_i w_{i'}/n_{i.}) = \tfrac{1}{2}\sum_{i\neq i'}\sum \left[\frac{w_i w_{i'}(n_{i.} - n_{i'.})}{n_{i.} n_{i'.}}\right]^2$$

$$= \tfrac{1}{2}\sum_{i\neq i'}\sum \left[\frac{e^2(n_{i.} - n_{i'.})}{(n_{i.}\alpha + e)(n_{i'.}\alpha + e)}\right]^2 = \tfrac{1}{2}\sum_{i\neq i'}\sum \left[\frac{en_{i.}(n_{i'.}\alpha + e) - en_{i'.}(n_{i.}\alpha + e)}{(n_{i.}\alpha + e)(n_{i'.}\alpha + e)}\right]^2$$

$$= \tfrac{1}{2}\sum_{i\neq i'}\sum (w_i - w_{i'})^2 \qquad = (a - 1)\sum_i w_i^2 - 2\sum_{i>i'}\sum w_i w_{i'}$$

$$= a\sum w_i^2 - (\sum w_i)^2 .$$

Hence in (34)

$$|T| = e^{-4}\left[a\sum w_i^2 - (\sum w_i)^2 + (n_{..} - a)\sum w_i^2\right] = e^{-4}\left[n_{..}\sum w_i^2 - (\sum w_i)^2\right] = e^{-4}D$$

as in (35).

Estimation of μ

$$\underline{1}'\underline{V}^{-1}\underline{y} = \sum_i(\Sigma \text{ elements in } \underline{y}_i'\underline{V}_i^{-1})$$

$$= \sum_i\sum_j\sum_k y_{ijk}(\Sigma \text{ elements in k'th row of j'th set of rows of } \underline{V}_i^{-1})$$

$$= \sum_i\sum_j\sum_k y_{ijk}(1/e + n_{ij}h_{i,jj} + \sum_{j'\neq j} n_{ij'}h_{i,jj'}), \text{ from (A12) and (A13)},$$

$$= \sum_i\sum_j\sum_k y_{ijk}/m_{ij}q_i \quad \text{from (A27)}$$

$$= \sum_i\sum_j n_{ij}\,\bar{y}_{ij.}/m_{ij}q_i$$

$$= \sum_i \frac{1}{q_i}\sum_j \frac{n_{ij}}{m_{ij}}\bar{y}_{ij.} \text{ as in (37).}$$

$$\underset{\sim}{1}'\underset{\sim}{V}^{-1}\underset{\sim}{1} = \sum_i (\Sigma \text{ all elements in } \underset{\sim}{V}_i^{-1})$$

$$= \sum_i \sum_j \sum_k (1/e + n_{ij}h_{i,jj} + \sum_{j' \neq j} n_{ij'}h_{i,jj'}) \text{ from above,}$$

$$= \sum_i \sum_j \sum_k \left[\frac{1}{m_{ij}} - \frac{n_{ij}\,\alpha}{m_{ij}^2\, q_i} - \frac{\alpha}{m_{ij}\, q_i} \left(\sum_{j'} \frac{n_{ij'}}{m_{ij'}} - \frac{n_{ij}}{m_{ij}} \right) \right] \text{, from (A23 and (A25)}$$

$$= \sum_i \left(\sum_j \frac{n_{ij}}{m_{ij}} \right) \left(1 - \frac{\alpha}{q_i} \sum_j \frac{n_{ij}}{m_{ij}} \right)$$

and using the definition of q_i this becomes

$$\underset{\sim}{1}'\underset{\sim}{V}^{-1}\underset{\sim}{1} = \sum_i \left(\sum_j \frac{n_{ij}}{m_{ij}} \right) \frac{1}{q_i} \quad \text{as in (38).}$$